CS 161
Summer 2018
7/19/18

Tutorial 4: Tree Algorithms
Graph Algorithms

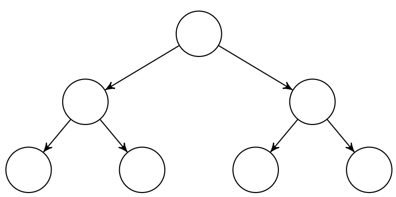
Tree Algorithms

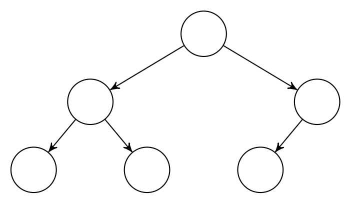
1. **(Fun with red-black trees) (Difficulty: Easy)**

   Recall the five rules for red-black trees:

   - Every vertex is colored **red** or **black**.
   - The root vertex is a **black** vertex.
   - A NIL child is a **black** vertex.
   - The child of a **red** vertex must be a **black** vertex.
   - For all vertices v, all paths from v to its NIL descendants have the same number of **black** vertices.

   Consider the following binary search trees. Which of the following vertices must be black in **any** valid red-black coloring?

(a) 

(b) 

2. **(Randomly built BST's) (Difficulty: Hard)**

Here, we prove that the average depth of a vertex in a randomly built binary search tree with n vertices is $O(log(n))$. A **randomly built binary search tree** with n vertices is one that arises from inserting the n keys in random order into an initially empty tree, where each of the $n!$ permutations of the input keys is equally likely.

Let $d(x, T)$ be the depth of vertex x in a binary tree T (the depth of the root is 0). Then, the average depth of a vertex in a binary tree T with n vertices is

$$\frac{1}{n}\sum_{x\in T} d(x, T)$$

.

(a) Let the *total path length* $P(T)$ of a binary tree T be defined as the sum of the depths of all vertices in T, so the average depth of a vertex in T with n vertices is equal to $\frac{1}{n}P(T)$. Show that $P(T) = P(T_L)+P(T_R)+n-1$, where T_L and T_R are the left and right subtrees of T, respectively.

[We are expecting: A rigorous mathematical proof.]

(b) Let $P(n)$ be the expected total path length of a randomly built binary search tree with n vertices. Show that

$$P(n) = \frac{1}{n}\sum_{i=0}^{n-1}(P(i) + P(n-i-1) + n - 1)$$

.

[We are expecting: A rigorous mathematical proof.]

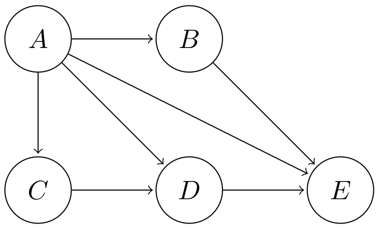
(c) Show that $P(n) = O(n\log(n))$. Hint: You might want to revisit quicksort.
[We are expecting: A short English justification.]

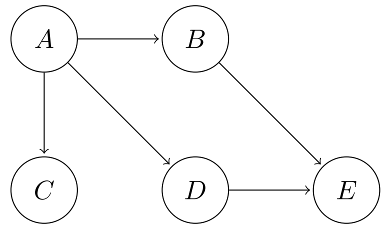
(d) Design a sorting algorithm based on randomly building a binary search tree. Show that its expected runtime is $O(n\log(n))$. Assume that a random permutation of n keys can be generated in $O(n)$-time.
[We are expecting: An English description of the algorithm and a brief justification for its runtime.]

Graph Algorithms

1. **(Fun with graphs) (Difficulty: Easy)**

   For each of the following graphs, assuming a vertex's neighbors will be visited in alphabetical order, what will be the ordering of end times from lowest to highest for `dfs` and order of visited times for `bfs`, starting from A.

   (a)
   

   (b)
   

2. **(DFS) (Difficulty: Easy)**

   Implement `dfs` without using recursion, still with runtime $O(|V| + |E|)$.

   [We are expecting: An pseudocode of the algorithm and a brief justification for its runtime.]

3. **(Robotic reduction) (Difficulty: Medium)**

   Consider a robot that can take steps of k distinct lengths $s_1, s_2, \ldots, s_k \in \mathbb{Z}$ in feet. The robot starts at position p_0 on a circular track with circumference length $T \in \mathbb{N}$ in feet and it starts walking counter-clockwise around the track to calibrate itself. It takes one step for each of the lengths $s_1, s_2, \ldots, s_k$, bringing it to position p_1.

   Describe an $O(T + kT)$-time algorithm that finds the fewest number of steps that the robot needs to take to get back to position p_0, assuming the robot can't turn around (all steps must be taken in the counter-clockwise direction).

   [We are expecting: An English description of the algorithm and a brief justification for its runtime.]

4. **(2SAT-SCC) (Difficulty: Hard)**

In 2SAT, you have a set of boolean clauses, each containing two variables. Your goal is to set all of these clauses to true, or report that doing so is not possible. For example,

$$(x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_3) \wedge (\bar{x}_2 \vee \bar{x}_3)$$

can be satisfied by setting x_1 and x_3 to true and x_2 to false.

(a) First, it helps to decompose clauses into their implications i.e. a clause represented by $a \vee b$ would be represented by $\bar{a} \rightarrow b$ and $\bar{b} \rightarrow a$. What are the implications in the clauses above?
[We are expecting: A list of implications.]

(b) To solve this problem with a graph algorithm, first built a graph by creating two vertices for each variable x: x and $\bar{x}$. Then, add directed edges based in the implications from each clause. What is the graphical representation of the above boolean clauses?
[We are expecting: A directed graph.]

(c) What do SCC's in these 2SAT graphs represent? What if x and $\bar{x}$ were part of the same SCC?
[We are expecting: Answers to the questions.]

(d) Assume we have a sink SCC C from the graph, and that no variable and its negation both appear inside of C. If C has vertices of the form $a, b, \ldots, z$, what corresponding component do we know must exist in the graph? Is there an edge between these two components?

[We are expecting: Answers to the questions.]

(e) Given this graph and the properties above, develop an algorithm to solve 2SAT.

[We are expecting: An English description of the algorithm.]